Journal of Statistical Physics, Vol. 27, No. 4, 1982

Spectral Analysis of a Nonlinear Oscillator Driven by Random and Periodic Forces. I. Linearized Theory

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Received August 11, 1981

We have combined the techniques of statistical and harmonic linearization to develop a linearized approximation theory for the calculation of the second-order statistics (i.e., autocorrelation functions and spectral densities) of nonlinear systems driven by both random and periodic forces. For the special case of a Duffing oscillator (a damped anharmonic oscillator with a cubic nonlinearity) driven by Gaussian white noise and by a sinusoidal force, explicit expressions for the renormalized (linearized) frequency, the autocorrelation function, and the spectral density of the oscillator displacement in terms of all the system parameters have been derived. We have determined the region of the parameter space in which the applied periodic force has a significant influence on the second-order statistics of the oscillator.

KEY WORDS: Nonlinear oscillator; statistical linearization; harmonic linearization; spectral densities.

1. INTRODUCTION

The anharmonic oscillator plays a fundamental role in the study of nonlinear systems. It is one of the simplest nonlinear systems that nevertheless exhibits features characteristic of more complicated ones.[(1)] Also, the anharmonic oscillator is frequently used as a model for nonlinear mechanical and electromagnetic systems.[(2)]

The anharmonic oscillator driven by a random force, i.e., the Duffing oscillator, is the simplest model for nonlinear systems subject to random excitations. The Duffing oscillator has therefore been used in a number of

This research was supported by the Office of Naval Research, by the National Science Foundation under grant No. CHE78-21460 and by a grant from Charles and Reneé Taubman.
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0022-4715/82/0400-0787$03.00/0

investigations dealing with approximation methods for the solutions of nonlinear stochastic dynamical processes.[3-6] One such approximate procedure is the method of statistical linearization,[4-7] a method that has proved to be very useful. Statistical linearization consists of the replacement of the nonlinear stochastic system by an "equivalent" linear one. The linear system is chosen so that the (suitably defined) error made by the replacement is minimized. We have applied the method of statistical linearization to a Duffing oscillator and have shown it to be a simple and effective method to obtain good approximate results for the autocorrelation function and spectral density of the oscillator for wide ranges of parameter values.[6] The results of an experimental investigation via analog circuits on the ranges of validity of statistical linearization for a Duffing oscillator will be published shortly.[8]

The anharmonic oscillator driven by a periodic force is one of the simplest nonlinear systems driven by a deterministic force. Approximate methods to deal with such systems have been extensively studied.[9] One of the simplest and most successful of these methods is that of harmonic linearization, developed by Krylov and Bogoliubov.[9] This method is applicable to systems driven by a periodic force and is based on the assumption that the response of the system occurs mainly at the driving frequency, with negligible contributions from higher harmonics.

It is clearly of interest to consider the response of nonlinear systems simultaneously driven by fluctuating and deterministic forces. Once again, the simplest case is the anharmonic oscillator simultaneously driven by a fluctuating force and a periodic one. To our knowledge, very little work has been done towards an analytic solution of this problem.[10] In this paper we develop a linearization (i.e., approximation) procedure that is a combination of the statistical and harmonic linearization techniques referred to above to obtain the autocorrelation function and spectral density of the linearized oscillator. We then analyze the relative effect of the deterministic and random forces on these measures of the oscillator response. The range of validity of these analytical results will be investigated via analog computer experiments now in progress.

The aim of this series of papers is to gain an understanding of and develop a methodology for analyzing the effects of an external, deterministic driving force, i.e., the "signal," on the observable properties of noisy, nonlinear systems. Ultimately, using these results we hope to be able to carry out an analysis of the "inverse problem," i.e., the determination of the form of the signal from the observed autocorrelation functions and/or spectral densities of the nonlinear system.

In Section 2 we briefly review the methods of statistical and harmonic linearization. Our method for the linearization of an anharmonic oscillator driven by both white noise and a sinusoidal force is developed in Section 3. The autocorrelation function and spectral density of the linearized oscillator are analyzed in Section 4. A discussion of the results is presented in Section 5.

2. REVIEW OF STATISTICAL AND HARMONIC LINEARIZATION

Consider a damped anharmonic oscillator of unit mass driven by a force $\mathcal{F}(t)$. The equation of motion for the oscillator displacement $x(t)$ is

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \beta f(x) = \mathcal{F}(t) \tag{2.1}$$

Here α is a damping parameter, ω_0 is the linear frequency, and $\beta f(x)$ is a nonlinear restoring force.

For arbitrary $\beta f(x)$ it is in general impossible to obtain an analytic solution to (2.1). One approach that has been used to obtain approximate solutions of the oscillator equation is to linearize it, i.e., to replace (2.1) by a harmonic oscillator of linear frequency γ,

$$\ddot{x} + \alpha\dot{x} + \gamma^2 x = \mathcal{F}(t) \tag{2.2}$$

The frequency γ is chosen in such a way that the error made by the replacement $\omega_0^2 x + \beta f(x) \to \gamma^2 x$ is minimized. The solution of (2.2) is then used as an approximate description of the oscillator (2.1).

The detailed procedure by which the coefficient γ^2 is determined depends on the nature of the driving force $\mathcal{F}(t)$. There exist well-established methods to choose the linear system that best represents the nonlinear one for two cases: when $\mathcal{F}(t)$ is a random force[3–7] and when it is periodic.[9] In this section we briefly review the linearization methods for these two cases.

2.1. Statistical Linearization

Consider the damped nonlinear oscillator (2.1) driven by a randomly fluctuating force $\mathcal{F}(t) \equiv F(t)$:

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \beta f(x) = F(t) \tag{2.3}$$

We will take $F(t)$ to be stationary, Gaussian distributed and delta-correlated with zero mean, i.e.,

$$\langle F(t) \rangle_F = 0 \tag{2.4a}$$

$$\langle F(t)F(t') \rangle_F = 2D\delta(t - t') \tag{2.4b}$$

The brackets $\langle\ \rangle_F$ in (2.4) denote an average over an ensemble of realizations of $F(t)$. The oscillator displacement is now also a random variable so that a solution of (2.3) must be interpreted statistically.

The error made in the dynamical equation when one replaces (2.1) by (2.2) is

$$\Delta(x) = \beta f(x) + \omega_0^2 x - \gamma^2 x \tag{2.5}$$

In the method of statistical linearization, γ is chosen so that the mean square error $\langle \Delta^2(x) \rangle_F$ is minimized at equilibrium, i.e., γ is the solution of the equation

$$\lim_{t\to\infty} \frac{\partial^2}{\partial \gamma^2} \langle \Delta^2(x) \rangle_F \equiv \frac{\partial^2}{\partial \gamma^2} \langle \Delta^2(x) \rangle = 0 \tag{2.6}$$

where we have introduced the definition

$$\lim_{t\to\infty} \langle g(x) \rangle_F \equiv \langle g(x) \rangle \tag{2.7}$$

It can be shown[4] that the choice (2.6) for γ ensures that the mean displacements $\langle x(t) \rangle$ of the linear and of the nonlinear oscillator are identical and that the difference in the mean square displacements $\langle x^2(t) \rangle$ for the two oscillators is minimized. This procedure is thus designed to replace a nonlinear system by the linear one that best reproduces the equilibrium mean and variance of the former.

From (2.5) in (2.6) one readily obtains the result

$$\gamma^2 = \omega_0^2 + \beta \frac{\langle x f(x) \rangle}{\langle x^2 \rangle} \tag{2.8}$$

The average indicated by the brackets in (2.8) is, as before, over an ensemble of realizations of $F(t)$ as $t \to \infty$ or, equivalently, over the equilibrium distribution of displacements of the oscillator. We denote this equilibrium distribution by $P(x)$ so that for an arbitrary function $g(x(t))$ we can write

$$\langle g(x) \rangle = \int_{-\infty}^{\infty} P(x) g(x)\, dx \tag{2.9}$$

The distribution $P(x)$ can in principle be obtained as follows. One constructs the Fokker–Planck equation for the conditional probability density $P(x, v, t \mid x_0, v_0)$ that the oscillator displacement $x(t)$ and velocity $\dot{x}(t)$ lie within dx of the value x and dv of the value v, respectively, given the initial values $x(0) = x_0$ and $\dot{x}(0) = v_0$.[10] The distribution $P(x)$ is the $t \to \infty$ limit of $P(x, v, t \mid x_0, v_0)$ integrated over all possible oscillator velocities. In practice, it is usually not possible to find the equilibrium distribution of nonlinear systems.[3] It then becomes necessary to introduce the further

[3] The exact equilibrium solution $P(x, v, \infty \mid x_0, v_0)$ *can* be found for the anharmonic oscillator. The effect of using this exact solution in the calculation of the averages is discussed in detail in Ref. 6.

approximation of using the equilibrium distribution $P_{\text{lin}}(x)$ of the already *linearized* problem to calculate the averages in (2.8). This distribution of course contains the frequency γ as a parameter so that (2.8) becomes an equation, usually transcendental, with the unknown frequency γ on both sides.

Once γ^2 has been determined, the second-order statistical properties of the linearized system (2.2) can be calculated and used as approximations to those of the nonlinear oscillator (2.1). In particular, the autocorrelation function of the linear oscillator displacement is readily found to be[11]

$$\langle x(t)x(t+\tau)\rangle = \frac{De^{-\alpha|\tau|/2}}{\alpha\gamma^2}\left(\cos\omega'|\tau| + \frac{\alpha}{2\omega'}\sin\omega'|\tau|\right) \tag{2.10}$$

where

$$\omega' = \left(\gamma^2 - \alpha^2/4\right)^{1/2} \tag{2.11}$$

The spectral density $S_{xx}(\omega)$ is the Fourier transform of the autocorrelation function, i.e.,

$$\begin{aligned} S_{xx}(\omega) &= \int_{-\infty}^{\infty} e^{i\omega\tau}\langle x(t)x(t+\tau)\rangle\, d\tau \\ &= \frac{2D}{\left(\gamma^2 - \omega^2\right)^2 + \alpha^2\omega^2} \end{aligned} \tag{2.12}$$

In an earlier paper in this series[6] we have carried out numerical calculations for nonlinear oscillators using the method described here and have compared the approximate results with more accurate ones obtained for the nonlinear oscillator. We have also analyzed systematic corrections to the linearization method. We find that statistical linearization leads to results for second-order statistics that are remarkably accurate over wide ranges of parameter values, a conclusion that is also borne out by the experimental results which will be published shortly.[8]

2.2. Harmonic Linearization

Consider the damped nonlinear oscillator (2.1) driven by a sinusoidal force $\mathfrak{F}(t) \equiv Q(t) = Q\sin\Omega t$,

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \beta f(x) = Q\sin\Omega t \tag{2.13}$$

The method of harmonic linearization is applicable to a system described by (2.13) when the parameter values are such that the long time response of the oscillator to a force of frequency Ω occurs mainly at that frequency, with negligible contributions to the oscillator displacement and velocity from higher harmonics. If this is the case, then one can write the approximate solution of (2.13) as

$$x(t) \approx a \sin(\Omega t - \theta) \tag{2.14}$$

where the amplitude a and the phase θ are to be determined. The nonlinear function $f(x)$ is expanded in a Fourier series in $(\Omega t - \theta)$ and only the first harmonic is retained, i.e.,

$$f(x) \approx q(a)\sin(\Omega t - \theta) = \frac{q(a)}{a} x \tag{2.15}$$

where the coefficient $q(a)$ is defined by the usual Fourier series relation

$$q(a) = \frac{1}{\pi} \int_0^{2\pi} f(a \sin u) \sin u \, du \tag{2.16}$$

In writing (2.15) we have assumed that $f(x)$ is an odd function of x. Equation (2.15) is a quasilinear approximation rather than a linear one, because $q(a)$ is in general a nonlinear function of the amplitude a.[4]

To determine the unknown parameters a and θ, the approximations (2.14) and (2.15) are substituted into the oscillator equation (2.13). The requirement that the resulting equation be identically satisfied leads to the following two nonlinear algebraic equations in the two unknowns:

$$\alpha a \Omega = Q \sin\theta \tag{2.17a}$$

$$(\omega_0^2 - \Omega^2)a + \beta q(a) = Q \cos\theta \tag{2.17b}$$

The solution of these equations clearly depends on the form of the quasilinearization coefficient $q(a)$, which in turn depends on the form of the nonlinearity $f(x)$.

Another way to view the procedure outlined above is in terms of the replacement of (2.1) by (2.2), i.e., the replacement of $\omega_0^2 x + \beta f(x)$ by the linear function $\gamma^2 x$. The frequency γ is found from (2.15) to be related to the amplitude a of (2.14) by

$$\gamma^2 = \omega_0^2 + \beta \frac{q(a)}{a} \tag{2.18a}$$

$$= \omega_0^2 + \beta \frac{\overline{x f(x)}}{\overline{x^2}} \tag{2.18b}$$

where the bar denotes a *time average* over one cycle of oscillation of the

[4] It should be noted that for more general nonlinearities [e.g., $F(x, \dot{x})$ in place of $f(x)$] the linearization procedure is more complicated since then the amplitude a and phase θ may be slowly varying functions of time. The quasilinear approximation to the nonlinearity is then also more complicated.

sinusoidal force, i.e.,

$$\overline{g(x)} \equiv \frac{1}{2\pi} \int_0^{2\pi} g(a \sin u)\, du \tag{2.19}$$

and where $x = a\sin(\Omega t - \theta)$ is the solution of the linearized problem. The statement (2.18b) of the result of harmonic linearization is designed to exhibit a formal similarity with the result (2.8) of statistical linearization. We stress that this similarity is only formal and not operational: whereas Eq. (2.8) is a prescription to obtain the linearization frequency, Eq. (2.18) is simply a formula relating γ^2 to the amplitude a, which must in turn be found from the prescription (2.17). Using (2.18a) in (2.17) thus gives a second relation between γ^2 and a:

$$a = \frac{Q}{\left[\alpha^2\Omega^2 + (\gamma^2 - \Omega^2)^2\right]^{1/2}} \tag{2.20}$$

Equations (2.18) and (2.20) together are sufficient to determine the linearization frequency γ.

The formal similarity between Eqs. (2.18b) and (2.8), although not operationally useful without the additional relation (2.20), can nevertheless be used to construct a formula for the linearization frequency of an oscillator driven by *both* noise and a periodic force. This case is considered in the next section.

3. LINEARIZATION OF AN ANHARMONIC OSCILLATOR SIMULTANEOUSLY DRIVEN BY A FLUCTUATING AND A PERIODIC FORCE

Let us now consider the anharmonic oscillator (2.1) driven by a randomly fluctuating force $F(t)$ and also by a sinusoidal force $Q\sin\Omega t$. The equation of motion is given by

$$\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \beta f(x) = F(t) + Q\sin\Omega t \tag{3.1}$$

The random force $F(t)$ is again taken to be stationary, Gaussian distributed, and delta correlated with zero mean, as in Eq. (2.4). We wish to replace (3.1) with a linear system driven by a combination of a fluctuating and a periodic force,

$$\ddot{x} + \alpha\dot{x} + \gamma^2 x = F(t) + Q\sin\Omega t \tag{3.2}$$

To determine the frequency γ we begin by postulating that Eqs. (2.8) and (2.18b) can be directly combined to apply to an oscillator driven by a combination of forces. This postulate leads to

$$\gamma^2 = \omega_0^2 + \beta \frac{\langle \overline{xf(x)} \rangle}{\langle \overline{x^2} \rangle} \tag{3.3}$$

where, as before, the brackets indicate an average over an ensemble of realizations of the random force as $t \to \infty$ and the bar indicates an average over a time interval $\Delta t = 1/\Omega$. In Eq. (3.3), x is the solution of the linearized problem (3.2), i.e.,

$$x(t) = x_F(t) + x_Q(t) \tag{3.4}$$

where $x_F(t)$ is the contribution to the displacement of the oscillator due to the random force $F(t)$ and

$$x_Q(t) = a \sin(\Omega t - \theta) \tag{3.5}$$

is the response to the periodic force. Note that the contribution x_F is stochastic while the sinusoidal contribution x_Q is deterministic. Substituting (3.4) into (3.2) and performing an ensemble average over the random force yields the equation

$$\ddot{x}_Q + \alpha \dot{x}_Q + \gamma^2 x_Q = Q \sin \Omega t \tag{3.6}$$

where we have used the fact that $\langle x_F(t) \rangle = 0$. Equation (3.6) is simply that of a linear oscillator driven by a sinusoidal force, i.e., precisely the one considered in Section 2.2. It can easily be seen that Eq. (2.20) relating the amplitude a in (2.14) to the linear frequency γ is therefore still valid for the amplitude a in (3.5). Equations (3.3) and (2.20) together are then sufficient to completely determine the linearization frequency. It should be noted that our prescription reduces to statistical linearization and to harmonic linearization, respectively, in the limits $F(t) \to 0$ and $Q \to 0$. This reduction to the proper limits lends credence to the postulate leading to Eq. (3.3).

To carry out our prescription in detail we choose a particular form for the nonlinearity in (3.1),

$$f(x) = x^3 \tag{3.7}$$

which corresponds to a Duffing oscillator. Equation (3.3) then yields

$$\begin{aligned} \gamma^2 &= \omega_0^2 + \beta \frac{\left\langle \overline{\left[x_F + a \sin(\Omega t - \theta) \right]^4} \right\rangle}{\left\langle \overline{\left[x_F + a \sin(\Omega t - \theta) \right]^2} \right\rangle} \\ &= \omega_0^2 + \beta \frac{\langle x_F^4 \rangle + 6a^2 \langle x_F^2 \rangle \overline{\sin^2(\Omega t - \theta)} + a^4 \overline{\sin^4(\Omega t - \theta)}}{\langle x_F^2 \rangle + a^2 \overline{\sin^2(\Omega t - \theta)}} \end{aligned} \tag{3.8}$$

where we have used the fact that $\overline{\sin^{2n+1}(\Omega t-\theta)}=0$. The ensemble averages $\langle x_F^n\rangle$ can be calculated using the solution of the Fokker–Planck equation for the linear oscillator equation

$$\ddot{x}_F+\alpha\dot{x}_F+\gamma^2 x_F=F(t) \tag{3.9}$$

The equilibrium distribution of the displacement x_F obtained from the Fokker–Planck equation is well known to be given by[11]

$$P_{\text{lin}}(x_F)=\left(\frac{\alpha\gamma^2}{2\pi D}\right)^{1/2}\exp\left(-\frac{\alpha\gamma^2}{2D}x_F^2\right) \tag{3.10}$$

Carrying out the ensemble and time averages indicated in (3.8) yields

$$\gamma^2=\omega_0^2+\beta\left[\frac{3D^2/\alpha^2\gamma^4+3a^2D/\alpha\gamma^2+3a^4/8}{D/\alpha\gamma^2+a^2/2}\right] \tag{3.11}$$

An equation involving only the unknown γ is finally obtained by eliminating the amplitude a from (3.11) with the use of (2.20). The resulting equation is

$$\gamma^2=\omega_0^2+\beta\left\{\frac{3D}{\alpha\gamma^2}+\frac{3Q^2}{\alpha^2\Omega^2+(\gamma^2-\Omega^2)^2}+\frac{3\alpha\gamma^2Q^4}{8D\left[\alpha^2\Omega^2+(\gamma^2-\Omega^2)^2\right]^2}\right\}$$
$$\times\left\{1+\frac{\alpha\gamma^2Q^2}{2D\left[\alpha^2\Omega^2+(\gamma^2-\Omega^2)^2\right]}\right\}^{-1} \tag{3.12}$$

Equation (3.12) must clearly be solved numerically. Once γ^2 is obtained, the autocorrelation function and spectral density for the linearized oscillator (3.2) can be calculated and used as an approximation to the corresponding functions for the nonlinear oscillator (3.1). We carry out this procedure for a variety of parameter values in Section 4.

Equation (3.12) for the linear frequency γ is the main result of this section. We wish to point out that it can also be obtained in an alternate but essentially equivalent manner to the one described above, by following a procedure that more closely parallels the steps of statistical linearization described in Section 2.1. One can construct a Fokker–Planck equation for the linear oscillator (3.2) which *includes* the periodic force. The solution of this Fokker–Planck equation is time dependent (oscillatory) even in the limit $t\to\infty$. We denote the velocity averaged solution in this limit by $P_{\text{lin}}(x,t)$ and recall that it is a function of γ. The averages in (3.3) can now be calculated using the prescription

$$\overline{g(x)} = \frac{\Omega}{2\pi}\int_0^{2\pi/\Omega} dt \int_{-\infty}^{\infty} dx\, g(x) P_{\text{lin}}(x,t) \tag{3.13}$$

This prescription applied to (3.3) with $f(x)$ given by (3.7) yields Eq. (3.12). We present the details of this method in Appendix A.

An important point should be noted about the linearization procedure presented above. Although the solution of the linearized oscillator equation is simply a superposition of the displacements caused by the random and deterministic forces [cf. Eq. (3.4)], these two forces are nonlinearly coupled through the linearization frequency γ. Equation (3.12) involves functions of the magnitudes of the forces and also the nonlinearity parameter β in a complicated, nonlinear fashion. Some of the effects of the nonlinear interactions are thus preserved through the linearization frequency.

4. RESULTS

In this section we analyze the behavior of the autocorrelation function and spectral density of the linearized oscillator (3.2) with frequency γ given by (3.12) for various ranges of parameter values.

We define the autocorrelation function for the oscillator displacement as

$$\begin{aligned} R_{xx}(\tau) &\equiv \langle \overline{x(t)x(t+\tau)} \rangle \\ &= \langle x_F(t)x_F(t+\tau)\rangle + \overline{x_Q(t)x_Q(t+\tau)} \end{aligned} \tag{4.1}$$

where x_F and x_Q are the contributions to the displacement due to the random and sinusoidal forces, respectively [cf. Eq. (3.4)]. Combining Eqs. (2.10), (2.20), and (3.5) we readily obtain

$$\begin{aligned} R_{xx}(\tau) &= \frac{De^{-\alpha|\tau|/2}}{\alpha\gamma^2}\left(\cos\omega'\tau + \frac{\alpha}{2\omega'}\sin\omega'|\tau|\right) \\ &\quad + \frac{Q^2/2}{\left[\alpha^2\Omega^2 + (\gamma^2-\Omega^2)^2\right]}\cos\Omega\tau \end{aligned} \tag{4.2}$$

where, as before, $\omega' = (\gamma^2 - \alpha^2/4)^{1/2}$. The spectral density corresponding to the autocorrelation function (4.2) is

$$\begin{aligned} S_{xx}(\omega) &= \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} R_{xx}(\tau) \\ &= \frac{2D + \pi Q^2/2\left[\delta(\omega+\Omega) + \delta(\omega-\Omega)\right]}{(\gamma^2-\omega^2)^2 + \alpha^2\omega^2} \end{aligned} \tag{4.3}$$

We begin by analyzing the behavior of the autocorrelation function (4.2). In particular, we wish to establish the conditions under which the sinusoidal force $Q \sin \Omega t$ in (3.1) affects the behavior of the autocorrelation function. At long times τ, $R_{xx}(\tau)$ becomes purely oscillatory due to the contribution of the sinusoidal force to the displacement, since the portion of $R_{xx}(\tau)$ associated with the random force vanishes. Thus, in this limit the behavior of $R_{xx}(\tau)$ is completely determined by the deterministic force. At shorter times $\tau \lesssim 2/\alpha$, however, the effect of the deterministic force on $R_{xx}(\tau)$ depends on the values of all the parameters. The sinusoidal force enters (4.2) in two ways: it contributes via (3.12) to the frequency ω', and it provides the term that oscillates at frequency Ω.

The relative contributions to γ and to $R_{xx}(\tau)$ of the sinusoidal and the random forces can be seen from (3.12) and (4.2) to be determined by the ratio

$$B \equiv \frac{Q^2 \alpha \gamma^2}{2D\left[\alpha^2\Omega^2 + (\gamma^2 - \Omega^2)^2\right]} \tag{4.4}$$

This is simply the ratio of the contributions to the mean square displacement of the oscillator due to the two driving forces, as may be seen by considering (4.2) at $\tau = 0$:

$$\langle \overline{x^2} \rangle = R_{xx}(0) = \frac{D}{\alpha\gamma^2} + \frac{Q^2}{2\left[\alpha^2\Omega^2 + (\gamma^2 - \Omega^2)^2\right]} \tag{4.5}$$

If Q is sufficiently small so that $B \ll 1$, then the sinusoidal force has a negligible effect on the effective frequency γ as well as on the correlation function $R_{xx}(\tau)$. In this case the problem reduces to statistical linearization, with γ as the solution of Eq. (3.12) in the absence of the Q-dependent terms, i.e.,[4,6]

$$\gamma^2 = \omega_0^2 + \frac{3D\beta}{\alpha\gamma^2} \tag{4.6}$$

The behavior of γ, $R_{xx}(\tau)$, and $S_{xx}(\omega)$ in this limit have been analyzed in detail elsewhere.[6]

If $B \gtrsim 1$, then the sinusoidal force affects the amplitude of $R_{xx}(\tau)$ at all times. The effect of the sinusoidal force on γ in this case depends on the magnitude of the nonlinearity parameter β. Since B in (4.4) contains the unknown frequency γ, further analysis is necessary to determine the conditions under which the sinusoidal force affects $R_{xx}(\tau)$ and γ, in terms of the other known parameters. To obtain these conditions, it is convenient to consider two regimes of behavior separately, i.e., at and off resonance. In all cases we shall consider only weak damping ($\alpha \ll \omega_0$) so that the oscillatory nature of the problem is retained even in the absence of a sinusoidal forcing function.

4.1. Resonance

A resonance condition occurs when the linear frequency γ and the forcing frequency Ω are equal. In this case all the terms in (4.2) oscillate at about the same frequency since $\alpha \ll \omega_0$ ensures that $\alpha \ll \gamma$ so that $\omega' \approx \gamma$. The contribution of the sinusoidal force then manifests itself as an addition to the amplitude of the oscillations of $R_{xx}(\tau)$ at the resonance frequency. To find the conditions of the parameters that will produce this behavior we set $\gamma = \Omega$ in Eq. (3.12) and solve the resulting equation for Ω. This yields the relation

$$\Omega^2 = \frac{\omega_0^2 + \left(\omega_0^4 + 4K\right)^{1/2}}{2} \tag{4.7}$$

where

$$K = \frac{3\beta D}{\alpha} \cdot \frac{\left[1 + Q^2/D\alpha + 1/8(Q^2/D\alpha)^2\right]}{\left[1 + 1/2(Q^2/D\alpha)\right]} \tag{4.8}$$

We note that for $\gamma = \Omega$ the ratio $Q^2/D\alpha$ occurring in (4.8) is simply twice the quantity B defined in (4.4). When $B \gtrsim 1$, then the contribution of the sinusoidal force to the amplitude $R_{xx}(\tau)$ is clearly important, as discussed earlier. Its effect on γ, however, depends on the magnitude of the nonlinearity parameter β. From (4.7) and (4.8), when $Q^2/D\alpha \gg 1$, we can see that if $\beta \ll \omega_0^4\alpha^2/3Q^2$ then the resonance condition (4.7) reduces to $\Omega = \gamma \approx \omega_0$. In this range of parameter values, $\gamma^2 \approx \omega_0^2 + \frac{3}{2}\beta\overline{\langle x^2\rangle}$ so that the above restriction on β is equivalent to the requirement that $\beta\overline{\langle x^2\rangle} \ll \omega_0^2$, i.e., that the nonlinearity not modify the restoring force. Thus in this regime the nonlinearity is sufficiently small that the oscillator is essentially linear for displacements $x \lesssim [\overline{\langle x^2\rangle}]^{1/2}$. Conversely, if $\beta \gg \omega_0^4\alpha^2/3Q^2$, then the resonance frequency is not the linear one but is, rather, determined by a combination of the parameters Q, β and $\alpha : \gamma^2 = \Omega^2 \approx (3\beta Q^2/\alpha^2)^{1/2}$. This frequency is seen to be independent of the magnitude of the random force and is determined only by the oscillator parameters and the deterministic force. These restrictions are shown in Fig. 1.

In Fig. 2 we have plotted the autocorrelation function $R_{xx}(\tau)$ vs. τ for a combination of parameters that leads to the resonance $\gamma = \Omega$. We have chosen parameters such that $Q^2/D\alpha = 4$ and $\beta \gg \omega_0^4\alpha^2/3Q^2$. On the same graph we show the autocorrelation function for the same parameter values but in the absence of the sinusoidal driving force ($Q = 0$). The effect of the

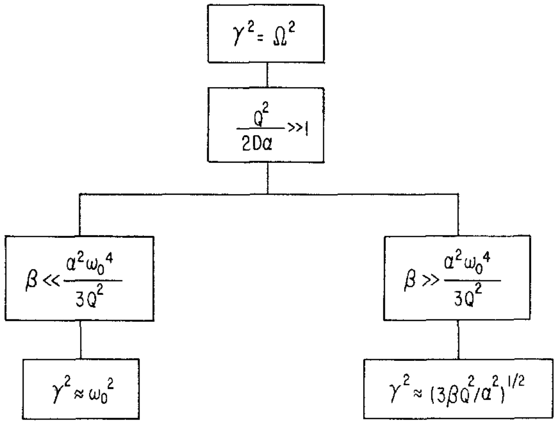

Fig. 1. Ranges of parameter values showing the effect of the sinusoidal force on γ in the case of the resonance $\gamma^2 = \Omega^2$.

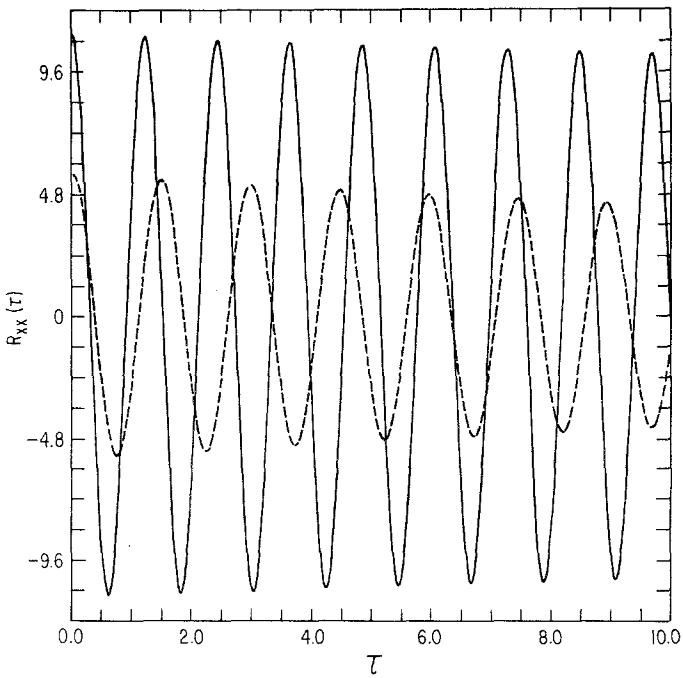

Fig. 2. Autocorrelation functions in the presence (solid curve) and absence (dashed curve) of a sinusoidal force. Resonance case. Solid curve parameters: $Q = 1$, $D = 5$, $\alpha = 0.05$, $\beta = 1$, $\omega_0^2 = 1$, $\Omega^2 = \gamma^2 = 26.96$. Dashed curve parameters: $Q = 0$, $\gamma^2 = 17.82$, otherwise same as above.

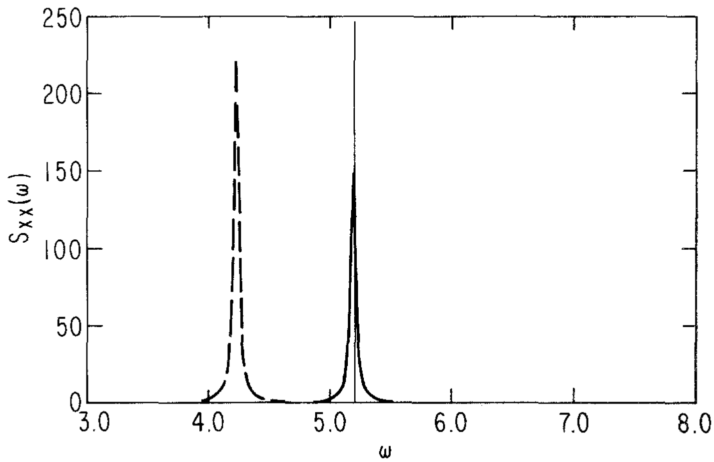

Fig. 3. Spectral densities corresponding to the autocorrelation functions of Fig. 2.

deterministic force is clearly apparent. The spectral densities $S_{xx}(\omega)$ for these two autocorrelation functions are plotted in Fig. 3 for $\omega > 0$. The delta function contribution is represented by a line at the resonant frequency $\omega = \Omega$.

4.2. Off Resonance

When $(\gamma^2 - \Omega^2)^2 \gg \alpha^2\Omega^2$, then the contribution to $R_{xx}(\tau)$ of the sinusoidal force relative to the random force becomes appreciable when $Q^2\alpha\gamma^2/2D(\gamma^2 - \Omega^2)^2 \gtrsim 1$, i.e., once again, when the quantity B in (4.4) is of $O(1)$ or larger. The frequency γ^2 can be obtained from (3.12) which, for $B \gg 1$, reduces to

$$\gamma^2 \approx \omega_0^2 + \frac{3\beta Q^2}{4(\gamma^2 - \Omega^2)^2} \tag{4.9}$$

To proceed further, we must distinguish two separate cases according to the relative magnitude of γ and Ω. If the frequency of the sinusoidal force is much higher than the linearized frequency $(\Omega^2 \gg \gamma^2)$ then (4.9) reduces to

$$\gamma^2 \approx \omega_0^2 + \frac{3\beta}{4}\frac{Q^2}{\Omega^4} \tag{4.10}$$

If $3\beta Q^2/4\Omega^4\omega_0^2 \ll 1$, then the linear oscillator is recovered $(\gamma^2 \approx \omega_0^2)$, i.e., the nonlinearity is too weak to modify the restoring force for displacements in the range $x \lesssim \langle x^2\rangle^{1/2}$.

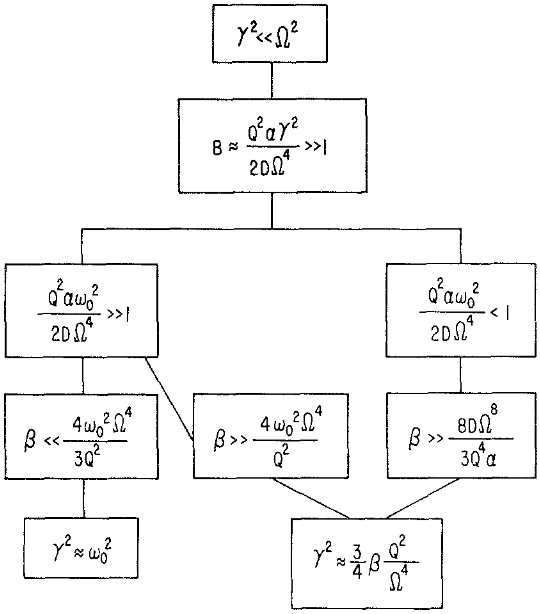

Fig. 4. Ranges of parameter values showing the effect of the sinusoidal force on γ for the off-resonance case $\gamma^2 \ll \Omega^2$.

If β and Q are sufficiently large, then the linearization frequency is determined by the sinusoidal force and the nonlinearity parameter according to $\gamma^2 \approx 3\beta Q^2/4\Omega^4$. To obtain this result for values of Q such that $Q^2\alpha\omega_0^2/2D\Omega^4 \gg 1$ requires that $\beta \gg 4\omega_0^2\Omega^4/3Q^2$. If $Q^2\alpha\omega_0^2/2D\Omega^4 < 1$, the requirement is that $\beta \gg 8D\Omega^8/3Q^4\alpha$. It can be seen that this second condition on β is more restrictive than the first, i.e., if the amplitude of the sinusoidal force is relatively small, then the nonlinearity parameter must be correspondingly larger in order for the sinusoidal force to affect the linearization frequency.

The conditions discussed in this subsection are summarized in Fig. 4. It must be stressed that there is an upper bound on the value of β for the analysis in the previous paragraph to be valid. This comes about because the off-resonance condition $\gamma^2 \ll \Omega^2$ implies that $\beta \ll 4\Omega^6/3Q^2$. If β does not satisfy this inequality, then the frequency of the applied force and the linearization frequency will approach one another and the appropriate analysis is that of Fig. 1. In Fig. 5 we present the autocorrelation function for a combination of parameters that leads to the behavior discussed above, i.e., $\Omega^2 \gg \gamma^2$, $\beta > 4\omega_0^2\Omega^4/3Q^2$. By comparing the amplitude and frequency of $R_{xx}(\tau)$ in the presence and absence of the sinusoidal force we indeed see

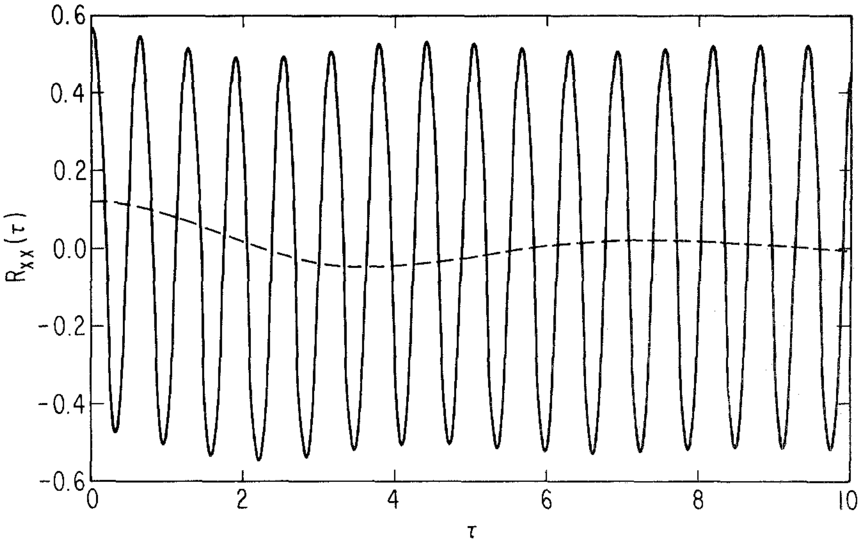

Fig. 5. Autocorrelation functions in the presence (solid curve) and absence (dashed curve) of a sinusoidal force. Off-resonance case. Solid curve parameters: $Q=100$, $D=0.05$, $\alpha=0.5$, $\beta=2$, $\omega_0^2=0.1$, $\Omega^2=100$, $\gamma^2=2.08 \ll \Omega^2$. Dashed curve parameters: $Q=0$, $\gamma^2=0.83$, otherwise same as above.

that this force completely modifies both aspects of the behavior of $R_{xx}(\tau)$ even though the system is far from resonance. The corresponding spectral densities $S_{xx}(\omega)$ are shown in Fig. 6.

When the frequency of the sinusoidal force is much lower than the linearized frequency ($\Omega^2 \ll \gamma^2$), we can further approximate (4.9) by

$$\gamma^2 \simeq \omega_0^2 + \frac{3\beta Q^2}{4\gamma^4} \tag{4.11}$$

If $3\beta Q^2/4\omega_0^6 \ll 1$ then the nonlinearity is too weak to affect the linear frequency, and the linear oscillator is recovered ($\gamma^2 \simeq \omega_0^2$).

If Q is sufficiently large so that $Q^2\alpha/2D\omega_0^2 \gg 1$ and β lies in the range $\omega_0^6/Q^2 \ll \beta \ll \alpha^3 Q^4/6D^3$, then the linearization frequency is determined by the sinusoidal force and by the nonlinearity parameter via the relation $\gamma^2 \simeq (\frac{3}{4}\beta Q^2)^{1/3}$. If β becomes larger than $\alpha^3 Q^4/6D^3$, then the sinusoidal force affects neither the amplitude of the autocorrelation function nor the frequency γ *within the approximations considered here*. The nonlinearity is then so large that the response of the oscillator to a low-frequency sinusoidal force manifests itself primarily at the higher harmonics of the driving

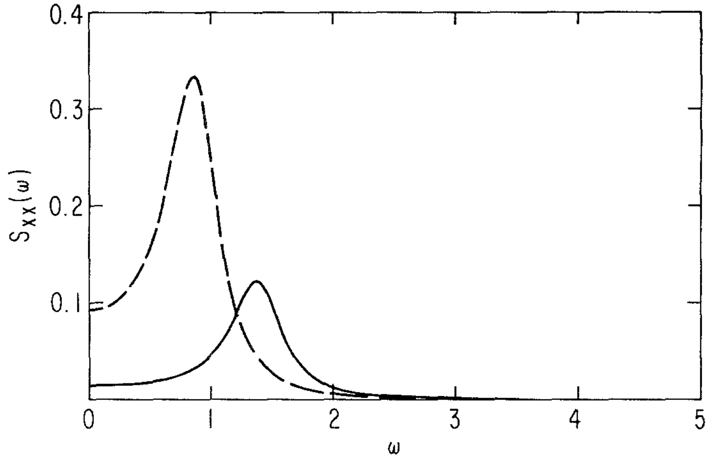

Fig. 6. Spectral densities corresponding to the autocorrelation functions in Fig. 5. The delta function contribution at $\omega = 10$ is off the scale of the figure and therefore not shown.

frequency Ω rather than at Ω itself. The entire linearization approach breaks down in this region of high nonlinearity.

The conditions discussed above are summarized in Fig. 7. It should be noted that the condition $\gamma^2 \gg \Omega^2$ implies a lower bound on β ($\beta \gg 4\Omega^6/3Q^2$) in order for the analysis in the previous paragraph to hold. In Figs. 8 and 9 we present autocorrelation functions and the corresponding spectral densities for a set of parameter values corresponding to the ranges in Fig. 7. Once again we see that for these choices the sinusoidal force indeed affects both the amplitude and frequency of the autocorrelation function, with corresponding effects on the spectral density.

5. CONCLUSION

We have developed a linearization method for nonlinear systems driven simultaneously by a random and a periodic force. The method is easy to implement and reduces to statistical linearization and to harmonic linearization, respectively, when either the random force or the periodic force is absent. We have applied our method to a damped anharmonic oscillator with a cubic nonlinearity and have found the "equivalent" linear oscillator that can then be used to obtain approximate second-order statistical quantities of the nonlinear oscillator. Our detailed analysis of the equivalent linear oscillator establishes ranges of parameter values in which

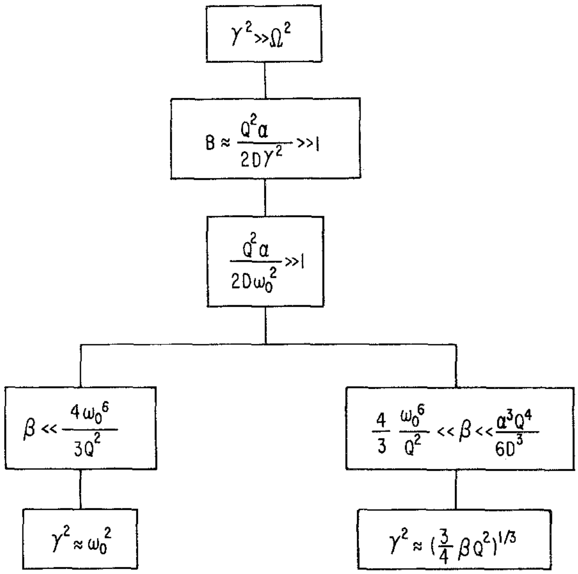

Fig. 7. Ranges of parameter values showing the effect of the sinusoidal force on γ for the off-resonance case $\gamma^2 \gg \Omega^2$.

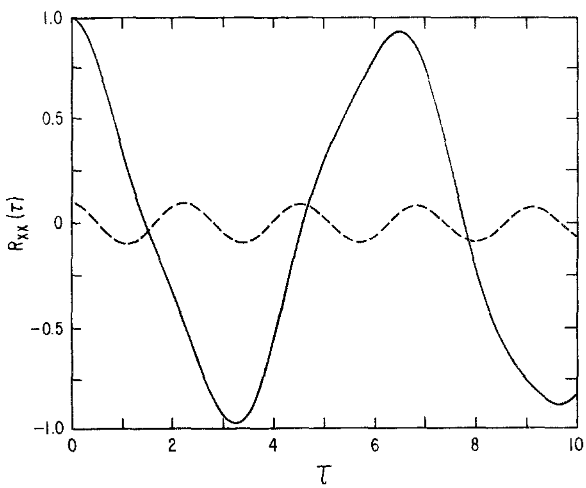

Fig. 8. Autocorrelation functions in the presence (solid curve) and absence (dashed curve) of a sinusoidal force. Off-resonance case. Solid curve parameters: $Q = 10$, $D = 0.05$, $\alpha = 0.05$, $\beta = 10$, $\omega_0^2 = 1$, $\Omega^2 = 1$, $\gamma^2 = 7.595 \gg \Omega^2$. Dashed curve parameters: $Q = 0$, $\gamma^2 = 6$, otherwise same as above.

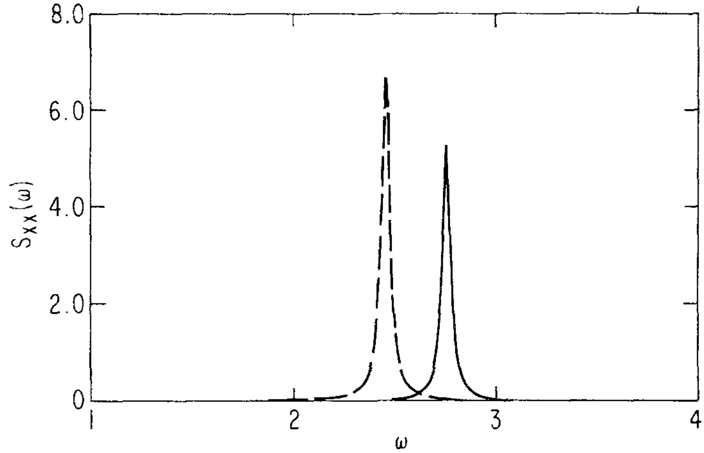

Fig. 9. Spectral densities corresponding to the autocorrelation functions in Fig. 8. The delta function contribution occurs at $\omega = 1$.

the applied periodic force appreciably affects the behavior of the autocorrelation function and spectral density of the noisy oscillator.

It is difficult to establish via analysis precise ranges of parameter values for which our procedure provides a good approximation to the second-order statistics of the nonlinear oscillator. We conjecture that our procedure is valid in those ranges of parameter values in which statistical and harmonic linearization are separately valid. It should be noted that the essentially nonperturbative nature of these linearization methods allows their applicability over rather wide ranges of the nonlinearity strength and of the amplitudes of the driving forces. We believe that our method may even be valid in some regimes where harmonic linearization alone may not be applicable because the random force may lead to a smoothing of the autocorrelation function. Thus, in the presence of the random force, some higher harmonic contributions that might cause harmonic linearization by itself to be invalid might in fact become unimportant. A detailed analysis of the validity of our method is presently being carried out via analog computer experiments.

APPENDIX A. FOKKER–PLANCK EQUATION FOR A LINEAR OSCILLATOR DRIVEN BY RANDOM AND PERIODIC FORCES

To construct the Fokker–Planck equation corresponding to the linear oscillator driven by a random and a periodic force, we rewrite (3.2) as a set of two first-order equations:

$$\dot{v} + \alpha v + \gamma^2 x = F(t) + Q \sin \Omega t \tag{A.1a}$$

$$\dot{x} = v \tag{A.1b}$$

We next introduce a complex mode amplitude $A(t)$ by

$$A = x + v/\lambda_- \tag{A.2}$$

where

$$\lambda_- \equiv \tfrac{1}{2}\alpha - i\omega' \tag{A.3}$$

and

$$\omega' \equiv (\gamma^2 - \alpha^2/4)^{1/2} \tag{A.4}$$

The mode amplitude $A(t)$ diagonalizes the equations of motion (A.1). Direct substitution of (A.2) into (A.1) yields

$$\dot{A} + \lambda_+ A = \lambda_-^{-1}[F(t) + Q \sin \Omega t] \tag{A.5}$$

where

$$\lambda_+ \equiv \lambda_-^* = \tfrac{1}{2}\alpha + i\omega' \tag{A.6}$$

The Fokker–Planck equation corresponding to (A.5) and its complex conjugate is found by standard methods[11] to be given by

$$\frac{\partial P(A, A^*, t)}{\partial t} = \left[\frac{\partial}{\partial A}\left(\lambda_+ A - \frac{Q}{\lambda_-} \sin \Omega t\right) + \text{c.c.} + D\left(\frac{1}{\lambda_-}\frac{\partial}{\partial A} + \frac{1}{\lambda_+}\frac{\partial}{\partial A^*}\right)^2 \right] P(A, A^*, t) \tag{A.7}$$

where c.c. stands for complex conjugate.

To solve Eq. (A.7) it is convenient to introduce the new variable

$$A' = [A + \phi(t)] \exp(\lambda_+ t) \tag{A.8}$$

where $\phi(t)$ is chosen to eliminate the drift term in the Fokker–Planck equation. Converting Eq. (A.7) to the variables (A', A'^*, t) and setting the coefficient of $\partial P/\partial A'$ equal to zero leads to

$$\phi(t) = \left[\phi(0) + \frac{\Omega Q}{\lambda_-(\Omega^2 + \lambda_+^2)}\right] \exp(-\lambda_+ t) - \frac{Q}{\lambda_-}\frac{\lambda_+ \sin \Omega t - \Omega \cos \Omega t}{\Omega^2 + \lambda_+^2} \tag{A.9}$$

The resulting equation is

$$\frac{\partial\chi(A',A'^*,t)}{\partial t}=D\left[\frac{e^{\lambda_+t}}{\lambda_-}\frac{\partial}{\partial A'}+\frac{e^{\lambda_-t}}{\lambda_+}\frac{\partial}{\partial A'^*}\right]^2\chi(A',A'^*,t)\quad\text{(A.10)}$$

where

$$\chi(A',A'^*,t)\equiv e^{-\alpha t}P(A',A'^*,t)\quad\text{(A.11)}$$

The full time-dependent solution of this equation conditional on the initial values $A_0',A_0'^*$ is[12]

$$\chi(A',A'^*,t)=\frac{1}{2\pi|\Delta|^{1/2}}\exp\left\{-\frac{1}{2\Delta}\left[a(A'-A_0')^2+2h(A'-A_0')(A'^*-A_0'^*)+a^*(A'^*-A_0'^*)^2\right]\right\}\quad\text{(A.12)}$$

where

$$a=D\frac{e^{2\lambda_-t}-1}{\gamma^2\lambda_-}\quad\text{(A.13a)}$$

$$h=-\frac{2D}{\gamma^2}(e^{\alpha t}-1)\quad\text{(A.13b)}$$

and

$$\Delta=|a|^2-h^2\quad\text{(A.13c)}$$

The probability distribution for the oscillator position and velocity as a function of time can now be obtained directly using the series of substitutions (A.11), (A.8) with (A.9), and (A.2) in (A.12). For the purposes of this paper we are only interested in the long time distribution of oscillator displacements. Performing the substitutions indicated above, taking the long time limit, and integrating over all oscillator velocities leads to the distribution

$$P_{\text{lin}}(x,t)=(\zeta/\pi)^{1/2}\exp\left\{-\zeta\left[x-\eta(t)/2\zeta\right]^2\right\}\quad\text{(A.14)}$$

where

$$\zeta=(\alpha/2D)\gamma^2\quad\text{(A.15a)}$$

and

$$\eta(t)=\frac{\alpha\gamma^2Q}{D\left[(\gamma^2-\Omega^2)^2+\alpha^2\Omega^2\right]}\left[(\gamma^2-\Omega^2)\sin\Omega t-\alpha\Omega\cos\Omega t\right]\quad\text{(A.15b)}$$

Use of (A.14) in (3.3) with the prescription (3.13) leads to the desired result (3.12).

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